

# Generating Specials: The Zorro Algorithm

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## Abstract

The concept of a configuration graph associated to a primitive, aperiodic substitution is introduced in [1] as a convenient graphical representation of the infinite indeterminism of the shift space of the substitution. The main result of [1] is an algorithm to calculate this graph from the substitution, in this paper we turn the tables and produce substitutions from graphs. We do this using the Zorro algorithm, an entirely constructive and easily applicable algorithm. In the process we show that any configuration graph can be obtained.

The first section contains standard definitions and the definition of configuration graphs. The second and third sections develop theory used in the proof of the algorithm as stated in section four. The algorithm is easily applied without knowledge of the underlying theory. Note that section three is nothing but a copy of results from [1] slightly modified to suit the present needs.

## 1 Preliminaries

### 1.1 Meeting Notational Needs

Let  $\mathcal{A}$  be any nonempty finite set of symbols, we call  $\mathcal{A}$  our *alphabet* and its members *letters*. By  $\mathcal{A}^*$  we understand the set of finite words constructed from the letters of  $\mathcal{A}$  including the empty word  $\epsilon$ . Equipped with the associative composition of concatenation,  $\mathcal{A}^*$  is the free monoid over  $\mathcal{A}$ . We furthermore let  $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\epsilon\}$  denote the set of nonempty words, and for any  $u \in \mathcal{A}^*$  we let  $|u|$  be the length of  $u$ , i.e., the number of letters of  $u$ . Given two words  $u$  and  $v$  of  $\mathcal{A}^*$  we say that  $u$  is a *factor* of  $v$  denoted  $u \dashv v$  if there exists  $w_1, w_2 \in \mathcal{A}^*$  with  $w_1 u w_2 = v$ .

We call members of  $\mathcal{A}^{\mathbb{Z}}$  (*two sided*) *sequences* over the alphabet  $\mathcal{A}$ . Let  $x$  be some sequence and let  $i \in \mathbb{Z}$ , we denote the letter at index  $i$  with  $x_{[i]}$ , given an additional  $j \in \mathbb{Z}$  with  $i \leq j$  we let  $x_{[i,j]}$  denote the word consisting of the letters from index  $i$  to index  $j$ , both included. We define the *language* of some sequence  $x$  to be the set  $\mathcal{L}(x) = \{\epsilon\} \cup \{u \in \mathcal{A}^* \mid \exists i, j \in \mathbb{Z}, i \leq j : u = x_{[i,j]}\}$  and call its members *factors* of  $x$ . We define the *shift*  $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$  by  $(\sigma(x))_{[i]} = x_{[i+1]}$  for  $x \in \mathcal{A}^{\mathbb{Z}}$  a sequence and  $i$  ranging over  $\mathbb{Z}$ . Elements of  $\mathcal{A}^{\mathbb{N}}$  are called *one sided sequences* over  $\mathcal{A}$ ; subscript notation and definition of language, factors

and shift apply to these as well, only the indices range over  $\mathbb{N}$  and not  $\mathbb{Z}$ . Note, however, that while the shift is bijective on  $\mathcal{A}^{\mathbb{Z}}$  it is only surjective on  $\mathcal{A}^{\mathbb{N}}$ .

Let  $u$  be any word of  $\mathcal{A}^*$  and  $x$  a one sided sequence, the concatenation  $ux$  is defined the obvious way. Given a two sided sequence  $x$  and  $i \in \mathbb{Z}$  we let  $x_{\infty, i}$  and  $x_{[i, \infty[}$  denote obvious one sided sequences. Given, on the other hand, any two single sided sequences  $x$  and  $y$ , we define the two sided sequence  $x.y$  by letting  $x.y_{[i]} = x[-i]$  for  $i < 0$  and  $x.y_{[i]} = y_{[i+1]}$  for  $i \geq 0$ , i.e., by reversing  $x$  and concatenating it with  $y$ , letting the first letter of  $y$  have index 0. We shall extend this notation in the obvious way to allow for finite words between the dot and the one sided sequences. For the sake of an example, let  $x$  and  $y$  be one sided sequences and let  $a$  be some letter, we then have that  $\sigma(x.ay) = xa.y$ .

By a *substitution*  $\tau$  we understand a map  $\tau : \mathcal{A} \rightarrow \mathcal{A}^+$ , it can be extended in the obvious way to a map respecting concatenation  $\tau : \mathcal{A}^* \rightarrow \mathcal{A}^*$ , furthermore to map single sided sequences to single sided sequences and by specifying  $\tau(x.y) = \tau(x).\tau(y)$  for any  $x, y \in \mathcal{A}^{\mathbb{N}}$  to map sequences to sequences; we shall not distinguish between a substitution and its extension. Note that for any  $u \in \mathcal{A}^*$  we have  $|\tau(u)| \geq |u|$  and that for any two substitutions  $\tau_1$  and  $\tau_2$  the composition  $\tau_1\tau_2$  defines a substitution as well.

## 1.2 Primitivity and Aperiodicity: Pretty Interesting Substitutions

In this subsection we introduce the concept of primitivity, the language associated with a substitution, the shift space associated with a substitution and finally the concept of aperiodicity. The different properties are easily verified if one proceeds in the order they are listed here.

**Definition 1** *A substitution  $\tau$  is said to be primitive if it holds that*

$$\exists n \in \mathbb{N} \forall a, b \in \mathcal{A} : b \vdash \tau^n(a)$$

*and that*

$$\exists a \in \mathcal{A} \forall N \in \mathbb{N} \exists n \in \mathbb{N} : |\tau^n(a)| > N.$$

Notice that the first of these properties implies the second if we have  $|\mathcal{A}| > 1$ , indeed the second property does nothing but exclude the substitution  $a \mapsto a$  in a theoretically convenient way.

**Proposition 2** *Let  $\tau$  be any primitive substitution. We have the following properties:*

- (i)  $\exists n \in \mathbb{N} \forall a, b \in \mathcal{A} \forall i \in \mathbb{N}_0 : b \vdash \tau^{n+i}(a)$
- (ii)  $\forall a \in \mathcal{A} \forall N \in \mathbb{N} \exists n \in \mathbb{N} : |\tau^n(a)| > N$
- (iii)  $\exists x \in \mathcal{A}^{\mathbb{Z}} \exists n \in \mathbb{N} : \tau^n(x) = x$

Now let  $\tau$  be some substitution, we define the *language* of  $\tau$  by

$$\mathcal{L}(\tau) = \{u \in \mathcal{A}^* \mid \exists a \in \mathcal{A} \exists n \in \mathbb{N} : u \vdash \tau^n(a)\}.$$

**Proposition 3** *Let  $\tau$  be any substitution. We have the following properties:*

- (i)  $\tau(\mathcal{L}(\tau)) \subseteq \mathcal{L}(\tau)$
- (ii)  $\forall u, v \in \mathcal{A}^* : u \dashv v, v \in \mathcal{L}(\tau) \Rightarrow u \in \mathcal{L}(\tau)$

*If furthermore  $\tau$  is primitive we get that:*

- (iii)  $\mathcal{A} \subseteq \mathcal{L}(\tau)$
- (iv)  $\forall n \in \mathbb{N} : \mathcal{L}(\tau) = \mathcal{L}(\tau^n)$

Consider now the non primitive substitution:

$$\tau_d : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 3.$$

We obviously have  $\mathcal{L}(\tau_d) = \{\epsilon, 2, 3\}$  and  $\mathcal{L}(\tau_d^2) = \{\epsilon, 3\}$  which demonstrates that primitivity is a necessary condition for the two lower properties.

We furthermore define the *shift space* associated with  $\tau$  by

$$X_\tau = \{x \in \mathcal{A}^{\mathbb{Z}} \mid \mathcal{L}(x) \subseteq \mathcal{L}(\tau)\}.$$

**Proposition 4** *Let  $\tau$  be any substitution. We have the following properties:*

- (i)  $\sigma(X_\tau) = X_\tau$
- (ii)  $\tau(X_\tau) \subseteq X_\tau$

*If furthermore  $\tau$  is primitive we get that:*

- (iii)  $\forall x \in X_\tau \forall u \in \mathcal{L}(x) \exists n \in \mathbb{N}_0 \forall i \in \mathbb{Z} : u \dashv x_{[i, i+n]}$
- (iv)  $\forall x \in X_\tau : \mathcal{L}(x) = \mathcal{L}(\tau)$
- (v)  $\forall n \in \mathbb{N} : X_\tau = X_{\tau^n}$
- (vi)  $X_\tau \neq \emptyset$

A sequence  $x \in \mathcal{A}^{\mathbb{Z}}$  is said to be *periodic* if there exists an  $n \in \mathbb{N}$  such that for all  $i \in \mathbb{Z}$  we have  $x_{[i]} = x_{[i+n]}$ ,  $n$  is called the length of the period. Finally let  $\tau$  be a primitive substitution. We say that  $\tau$  is *periodic* if  $X_\tau$  is finite. This is equivalent to  $\tau$  having a periodic member of  $X_\tau$  which is again equivalent to having all members of  $X_\tau$  periodic. *Aperiodicity* is obviously defined as the lack of periodicity for sequences as well as substitutions.

We end this somewhat tedious subsection with a small but handy lemma:

**Lemma 5** *Let  $\tau$  be a primitive, aperiodic substitution and let  $x \in X_\tau$ . We have that*

$$\forall i, j \in \mathbb{Z} : x_{[i, \infty[} = x_{[j, \infty[} \Leftrightarrow i = j$$

The proof is an easy application of the definitions above, a symmetrical version of the lemma also holds.

### 1.3 Orbit classes, specials and configuration graphs

Let  $\tau$  be a primitive, aperiodic substitution. By definition this implies that  $X_\tau$  is infinite. In this subsection we shall consider the structure of  $X_\tau$ , in particular we shall present the concept of a configuration graph associated to  $\tau$  which is a convenient graphical representation of the infinite indeterminism of  $X_\tau$ .

**Definition 6** *Let  $\tau$  be any substitution. Let  $x, y \in X_\tau$ . We define the following relations:*

- (i)  $x \sim_o y \Leftrightarrow \exists m \in \mathbb{Z} \forall i \in \mathbb{Z} : x_{[i]} = y_{[i+m]}$
- (ii)  $x \sim_r y \Leftrightarrow \exists m \in \mathbb{Z} \exists M \in \mathbb{Z} \forall i \geq M : x_{[i]} = y_{[i+m]}$
- (iii)  $x \sim_l y \Leftrightarrow \exists m \in \mathbb{Z} \exists M \in \mathbb{Z} \forall i \leq M : x_{[i]} = y_{[i+m]}$

We name these relations *orbit equivalence*, *right tail equivalence* respectively *left tail equivalence* and immediately verify that they are indeed equivalence relations. The equivalence classes under orbit equivalence are called *orbit classes* and since both right and left tail equivalence respect orbit equivalence they define equivalence relations on the orbit classes as well.

**Definition 7** *Let  $\tau$  be any substitution. A sequence  $x \in X_\tau$  is called left special if there exists  $y \in X_\tau$  with*

$$x_{[-1]} \neq y_{[-1]} \quad x_{[0,\infty[} = y_{[0,\infty[}.$$

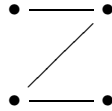
*An orbit class  $C \in X_\tau / \sim_o$  is called left special if there exists an orbit class  $D \in X_\tau / \sim_o$  with  $C \neq D$  and  $C \sim_r D$ .*

And yes, an easy application of lemma 5 shows that if  $\tau$  is primitive and aperiodic then an orbit class is left special if and only if it contains a left special sequence. *Right special* sequences and orbit classes are defined symmetrically.

As mentioned in theorem 1.5 of [1] the number of left as well as right special orbit classes is finite but nonzero if  $\tau$  is primitive and aperiodic. This makes the following definition meaningful:

**Definition 8** *Let  $\tau$  be a primitive, aperiodic substitution. The configuration graph is a bipartite graph defined as follows: The set of left vertices are the equivalence classes of orbit classes under left tail equivalence that contain a special orbit class. The set of right vertices are defined symmetrically and each special orbit class gives rise to an edge connecting the left and right equivalence classes that contain it.*

As an example, the primitive, aperiodic substitution  $1 \mapsto 121$ ,  $2 \mapsto 2112$  has the following configuration graph:



The calculation of configuration graphs is by no means a trivial exercise, indeed an algorithm doing this is the main result of [1]. This algorithm is most conveniently implemented online, see [2] for details.

## 2 Generators

**Definition 9** Let  $\tau$  be any substitution. Let  $(v, u, w) \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+$ . We say that  $(v, u, w)$  is a generator for  $\tau$  if  $u \in \mathcal{L}(\tau)$  and furthermore  $\tau(u) = vuw$ . We denote by  $G_\tau$  the set of all generators for  $\tau$ .

Given a generator  $(v, u, w)$  we shall refer to  $v$ ,  $u$  and  $w$  as the *left wing*, the *center* respectively the *right wing* to facilitate the language. Furthermore we shall refer to the length of the center as the length of the generator.

**Definition 10** Let  $\tau$  be any substitution and let  $(v, u, w) \in G_\tau$ . We define the completion of  $(v, u, w)$  by

$$(v, u, w)^* = \cdots \tau^2(v) \tau(v) v u . w \tau(w) \tau^2(w) \cdots$$

and note that this is a member of  $X_\tau$ .

This definition is our main justification for working with generators: they provide a means of creating members  $X_\tau$  and they do so in a nice way as we shall see below. But before we start completing let us first impose some structure on the set of generators.

**Definition 11** Let  $\tau$  be any substitution and let  $(v, u, aw)$  be a generator with  $v, u \in \mathcal{A}^+$ ,  $a \in \mathcal{A}$  and  $w \in \mathcal{A}^*$ . Then obviously  $(v, ua, w\tau(a))$  is a generator as well and we say it is constructed from the original by right extension; left extension is defined similarly. We say that two generators  $g_1$  and  $g_2$  for  $\tau$  are  $G$  related (denoted by  $g_1 \sim_G g_2$ ) if there exists a generator  $g_3$  such that  $g_3$  can be constructed from  $g_1$  by a series of (possibly zero) right and left extensions and  $g_3$  can be constructed similarly from  $g_2$ .

One quickly realizes that left as well as right extensions are deterministic, i.e., any generator can be left or right extended in exactly one way. Furthermore, right and left extensions are independent since they take place on different sides of the center, so to speak, and this implies that their order can be exchanged in a series of mixed extensions. Summing up, the relation defined above is transitive as well as obviously reflexive and symmetric, i.e., it is an equivalence relation.

**Definition 12** Let  $\tau$  be any substitution. We define the basic generators to be all generators that are not  $G$  related to any shorter generator.

We shall see shortly that there is exactly one basic generator in each equivalence class. But let us pause to consider how we would calculate the basic generators of a substitution, this turns out to be very easy in the case of primitive substitutions:

**Lemma 13** Let  $\tau$  be any substitution and let  $g = (v, aub, w)$  be any generator of two or more letters with  $v, w \in \mathcal{A}^+$ ,  $u \in \mathcal{A}^*$  and  $a, b \in \mathcal{A}$ . It is basic if and only if  $|\tau(a)| > |v|$  and  $|\tau(b)| > |w|$ .

*Proof:* Suppose one of the length inequalities fail, say,  $|\tau(a)| \leq |v|$ . Then we can write  $v = \tau(a)v'$  for some  $v' \in \mathcal{A}^*$  and  $(v'a, ub, w)$  is a generator shorter than  $g$  and obviously  $G$  related to  $g$ .

Now suppose both length inequalities hold. Let  $n \in \mathbb{N}_0$ . We shall show by complete induction on  $n$  that if  $g'$  and  $g''$  are two more generators and  $g$  can be extended to  $g''$  in a series of  $n$  extensions and  $g'$  can be extended to  $g''$  in another series of extensions, then  $g'$  is longer than or has the same length as  $g$ . Let  $m$  be the number of left extensions of the  $n$  steps and let  $m'$  be the number of left extensions in the steps extending  $g'$  to  $g''$ . If both are nonzero we can remove one left extension from both series and still end up with a common result, since left and right extensions commute, and afterwards apply the inductive hypothesis. We cannot have  $m = 0$  and  $m' > 0$  since the first would let the left length inequality hold for  $g''$  and the second would contradict this. This leaves us with  $m \geq m'$  and since the same arguments applies to right extensions we have finished our inductive argument and the proof.  $\square$

**Corollary 14** *Let  $\tau$  be a primitive substitution. The following holds:*

- (i) *All one letter generators are basic.*
- (ii) *Let  $(v, ab, w)$  be any two letter generator with  $v, w \in \mathcal{A}^+$  and  $a, b \in \mathcal{A}$ . It is basic if and only if  $\tau(a) = va$  and  $\tau(b) = bw$ .*
- (iii) *No generators of three or more letters are basic.*

Notice that the primitivity condition is necessary for part (iii) since a non primitive substitution may have basic generators of any length. Consider for instance the following non primitive substitution:

$$0 \mapsto 01230, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 30123.$$

This has the generator  $(0123, 0123, 0123)$  which is basic by the lemma thus contradicting the corollary.

The set of basic generators of a primitive substitution is very easily calculated using the corollary: The one letter generators can be read off the definition of the substitution directly; the two letter generators in question are all those that can be constructed from mating a one letter empty right wing "generator" with a one letter empty left wing "generator", bearing in mind that the center must always be in  $\mathcal{L}(\tau)$ . As an example consider the following primitive substitution:

$$0 \mapsto 042, 1 \mapsto 142, 2 \mapsto 042, 3 \mapsto 043, 4 \mapsto 01432.$$

This has the four basic generators  $(01, 4, 32)$ ,  $(04, 20, 42)$ ,  $(04, 21, 42)$  and  $(04, 30, 42)$  and no more, in particular  $(04, 31, 42)$  is not even a generator.

The following proposition justifies the basic generators as being, in essence, all generators:

**Proposition 15** *Let  $\tau$  be any substitution. We then have:*

- (i) *No two different basic generators are  $G$  related.*

(ii) Any generator is  $G$  related to a unique basic generator.

*Proof:* The proof of (i) proceeds similarly to the proof of the second part of the lemma, i.e., complete induction on the number of steps required to extend  $g$  to some generator that another basic generator can be extended to as well. Common left extensions are handled by the inductive hypothesis and left extensions in only one of the extension series are contradicted by the lemma. The proof of (ii) is immediate by induction on the length of the generator by the definition of basic generators; the uniqueness is a spinoff from part (i).  $\square$

It is now time to consider how these structures on  $G_\tau$  interact with the completion of members of  $G_\tau$ . The following result is a pretty one:

**Proposition 16** *Let  $\tau$  be a primitive, aperiodic substitution and let  $g_1, g_2 \in G_\tau$ . We have that*

$$g_1^* \sim_o g_2^* \iff g_1 \sim_G g_2.$$

*Proof:* The arrow leading left is immediate since left and right extension preserve completion up to orbit equivalence.

Assume now that  $g_1^* \sim_o g_2^*$ . Assume initially that  $g_1^* = g_2^*$ . Let  $n_1, n_2 \in \mathbb{N}$  be the length of the right wing of  $g_1$  respectively  $g_2$ . Since

$$\sigma^{-n_1}(\tau(g_1^*)) = g_1^* = g_2^* = \sigma^{-n_2}(\tau(g_2^*)) = \sigma^{-n_2}(\tau(g_1^*))$$

aperiodicity ensures that  $n_1 = n_2$ . This immediately implies that if  $g_1$  and  $g_2$  are of equal length then they are equal, and if they are not, then the shorter can be left extended to obtain longer. If  $g_1^* \neq g_2^*$  then there must exist a  $p \in \mathbb{Z}, p \neq 0$  such that  $\sigma^p(g_1^*) = g_2^*$ . In case  $p > 0$  then by performing  $p$  right extensions of  $g_1$  we are in the situation above. The case  $p < 0$  is handled by right extending  $g_2$ .  $\square$

With the construction of specials in mind, the following result is promising:

**Proposition 17** *Let  $\tau$  be a primitive, aperiodic substitution and let  $g_1, g_2 \in G_\tau$ . We have that  $g_1^* \sim_r g_2^*$  holds if and only if there exist two generators  $g'_1 \sim_G g_1$  and  $g'_2 \sim_G g_2$  with identical right wings.*

*Proof:* Assume that  $g_1^* \sim_r g_2^*$  holds. If we have the luck that  $g_1 \sim_G g_2$  then by definition there exists a  $g'$  with  $g_1 \sim_G g'$  and  $g_2 \sim g'$  and letting  $g'_1 = g'$  and  $g'_2 = g'$  concludes the case. If, on the other hand,  $g_1 \not\sim_G g_2$  holds then we have the existence of  $p, j \in \mathbb{Z}$  such that

$$\forall i \geq j : \sigma^p(g_1^*)_{[i]} = g_2^*_{[i]}$$

and

$$\sigma^p(g_1^*)_{[j-1]} \neq g_2^*_{[j-1]}.$$

Assume initially that  $p = 0$ . If we further assume assume that  $j \leq 0$ , then we can halfway duplicate the calculations from the proof of proposition 16: Let  $n_1, n_2 \in \mathbb{N}$  be the length of the right wing of  $g_1$  respectively  $g_2$ . We now get:

$$\begin{aligned} \sigma^{-n_1}(\tau(g_1^*))_{[n_2, \infty[} &= g_1^*_{[n_2, \infty[} \\ &= g_2^*_{[n_2, \infty[} \\ &= \sigma^{-n_2}(\tau(g_2^*))_{[n_2, \infty[} \\ &= \sigma^{-n_2}(\tau(g_1^*))_{[n_2, \infty[} \end{aligned}$$

This by lemma 5 is enough to ensure that  $n_1 = n_2$  which proves that the two generators have identical right wings. Now if  $j > 0$  then we perform  $j$  right extensions on both generators and proceed as above, this concludes the case  $p = 0$ . And as above, if  $p > 0$  then we do  $p$  right extensions of  $g_1$ , if  $p < 0$  then we do  $p$  right extensions of  $g_2$  and in both cases proceed as in the case  $p = 0$ . The reverse is immediate.  $\square$

Given two basic generators  $g_1$  and  $g_2$  with  $g_1 \approx_G g_2$  and suppose we'd like to know whether  $g_1^* \sim_r g_2^*$ . The proposition above tells us to look for  $G$  related generators with identical right wings, but this is not an algorithmically very pleasant task. But the proof above shows that  $g_1'$  and  $g_2'$  – if they exist at all – can be constructed by doing nothing but right extensions of  $g_1$  respectively  $g_2$ . After possibly undoing some pairwise identical right extensions we can furthermore obtain generators with identical right wings that disagree on either their rightmost letter of the center or the letter just before that. If now additionally  $\tau$  is regular, then this puts a maximum limit to the length of the desired common right wing, thereby making the test for  $g_1^* \sim_r g_2^*$  a finite story. Let us list an even simpler and most useful case:

**Corollary 18** *Let  $\tau$  be a primitive, aperiodic, postfix free substitution and let  $g_1, g_2 \in G_\tau$  with  $g_1 \approx_G g_2$ . We have that  $g_1^* \sim_r g_2^*$  holds if and only if the right wings of  $g_1$  and  $g_2$  are identical.*

A final note to conclude this section: The definition of the completion of a generator is not entirely symmetrical with respect to the left and right wings of the generator. The given definition has the pleasant property that right extending the generator shifts the completion one step; we rely heavily on this in the proofs above. On the other hand, one might fear that this would introduce some asymmetry to completions. This, however, is not the case as long as we stick to orbit classes. Indeed, the symmetrical versions of both proposition 17 and corollary 18 above hold, this is most easily checked by shifting to opposite substitutions.

### 3 Generating specials

**Definition 19** *Let  $\tau$  be any substitution. The leftmost letter graph (the ll graph) is defined to be the graph with the letters of  $\mathcal{A}$  as vertices and with one directed edge leaving each vertex  $a \in \mathcal{A}$  arriving at the leftmost letter of  $\tau(a)$ . The rightmost letter graph (the rl graph) is defined similarly.*

**Definition 20** *Let  $\tau$  be any substitution and let  $n \in \mathbb{N}$ . We say that  $n$  is a left segregating number if for any two words  $u, v \in \mathcal{L}_n(\tau)$  with differing leftmost letter we have that the length of the common prefix of  $\tau(u)$  and  $\tau(v)$  is less than or equal to  $\min\{|\tau(u)|, |\tau(v)|\} - n$ . Right segregating numbers are defined similarly.*

Note that not all substitutions have a segregating numbers. Consider for instance the following primitive, aperiodic substitution:

$$\tau_e : a \mapsto c, b \mapsto c, c \mapsto db, d \mapsto ca.$$



Squaring this we get a substitution with the two generators  $(d, bca, cdb)$  and  $(ca, cdb, cdb)$ . This implies that for any  $n \in \mathbb{N}$  there exists  $u \in A^*$  with  $|u| = n-1$  and  $au, bu \in \mathcal{L}_n(\tau_e)$  which shows that  $n$  cannot be a left segregating number since we have that  $\tau_e(au) = \tau_e(bu)$ . On the other hand, note that for any prefix free substitution 1 will do as left segregating number, similarly any postfix free substitution has 1 as right segregating number. We say that a substitution is *segregating* if it has both a left and a right segregating number. As is often the case, regular substitutions behave nicely:

**Proposition 21** *Let  $\tau$  be any primitive, regular substitution. Then  $\tau$  is segregating.*

*Proof:* We prove only the existence of the left segregating number, the right case is symmetrical. Since  $\tau$  is primitive there must exist an  $a \in \mathcal{A}$  with  $\tau(a) > 1$ . By minimality there exists an  $s \in \mathbb{N}$  such that any  $u \in \mathcal{L}_s(\tau)$  contains  $a$ . Now let

$$P = \sum_{a \in \mathcal{A}} |\tau(a)|, \quad Q = \max_{a \in \mathcal{A}} |\tau(a)|.$$

It now follows from theorem 1.6 in [3] that  $s(P - |\mathcal{A}| + Q - 1)$  is a left segregating number.  $\square$

**Definition 22** *Let  $\tau$  be any substitution with a left segregating number. Let  $n \in \mathbb{N}$  be the least such. We define the left segregating graph (the ls graph) as follows: The vertices are all pairs of words from  $\mathcal{L}_n(\tau)$  which differ at their leftmost letter. One directed edge leaves each vertex, if the vertex is  $(u, v)$  then the destination is obtained by removing the common prefix from  $\tau(u)$  and  $\tau(v)$  and reading the leftmost  $n$  letters from each remaining word. The right segregating graph (the rs graph) is defined similarly for a substitution with a right segregating number.*

It is time for an example, consider the following primitive, aperiodic, regular substitution:

$$\tau_4 : 0 \mapsto 10, 1 \mapsto 0.$$

The ll and rl graphs are as follows:

$$\text{ll : } \quad 0 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \quad \text{rl : } \quad \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} 0 \quad 1.$$

As left segregating number 1 will do, and clearly it is the least such. On the other hand, 2 is the least right segregating number. Since  $\mathcal{L}_1(\tau) = \{0, 1\}$  and  $\mathcal{L}_2(\tau) = \{00, 01, 10\}$  we get the following ls and rs graphs:

$$\begin{array}{ll} \text{ls : } & (0, 1) \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (1, 0) & \text{rs : } & \begin{array}{ccc} (00, 01) & & (10, 01) \\ & \searrow \quad \nearrow & \\ (01, 00) & & (01, 10) \end{array} \end{array}$$

We say that any of the graphs defined above are *subfixed* if for each vertex  $v$ ,  $v$  either loops to itself (i.e., the edge leaving  $v$  goes back to  $v$ ) or the edge

leaving  $v$  goes to some other vertex that loops to itself. Of the graphs in the example above only the  $rl$  graph is subfixed. It is, however, the case that for any segregating substitution  $\tau$  there exists an  $n \in \mathbb{N}$  such that all the graphs  $ll$ ,  $rl$ ,  $ls$  and  $rs$  for  $\tau^n$  are subfixed. To realize this, notice first that if  $\tau$  is segregating then so is any nonzero power of  $\tau$ . Then note that raising the power of  $\tau$  by one corresponds to extending each edge by its immediate successor in any of the graphs above. Finally let  $m$  be the least common multiple of the length of all cycles in all the graphs (each must have at least one cycle if  $\tau$  is primitive and aperiodic). Then raising  $\tau$  to the power of any positive multiple of  $m$  ensures that all vertices that are in cycles the original graph now loop to themselves and by choosing a sufficiently high multiple we can make all other vertices connect to one of these vertices. In the simple example above choosing  $n = 2$  will work, i.e., for  $\tau_4^2$  all the graphs  $ll$ ,  $rl$ ,  $ls$  and  $rs$  are subfixed. The following theorem is our main justification for this as well as the preceding section:

**Theorem 23** *Let  $\tau$  be any primitive, aperiodic, segregating substitution with all the graphs  $ll$ ,  $rl$ ,  $ls$  and  $rs$  subfixed. Then for any left or right special sequence  $u \in X_\tau$  there exists a generator  $g \in G_\tau$  such that  $g^* \sim_o u$ .*

To prove this, consider first the following lemma:

**Lemma 24** *Let  $\tau$  be any primitive, aperiodic substitution with a right segregating number and with the  $rs$  graph subfixed. Suppose we have  $u, v \in X_\tau$  with  $u_{[0, \infty[} = v_{[0, \infty[}$  and  $u_{[-1]} \neq v_{[-1]}$ . Then there exist  $u', v' \in X_\tau$  with  $u'_{[0, \infty[} = v'_{[0, \infty[}$  and  $u'_{[-1]} \neq v'_{[-1]}$  and*

$$u[-n, -1] = u'[-n, -1], \quad v[-n, -1] = v'[-n, -1]$$

and

$$u = \sigma^{-p}(\tau(u')), \quad v = \sigma^{-p}(\tau(v')),$$

where  $n \in \mathbb{N}$  is the least right segregating number and  $p \in \mathbb{N}_0$  is the length of the common postfix of  $\tau(u'[-n, -1])$  and  $\tau(v'[-n, -1])$ .

*Proof of lemma:* By corollary 12 of [4] there exists  $x, y \in X_\tau$  with  $u \sim_o \tau(x)$  and  $v \sim_o \tau(y)$ . By lemma 3.1 of [1] we get that  $x \sim_r y$ . But since  $u \approx_o v$  we also have  $x \approx_o y$  and we may choose  $u' \sim_o x$  and  $v' \sim_o y$  with  $u'_{[0, \infty[} = v'_{[0, \infty[}$  and  $u'_{[-1]} \neq v'_{[-1]}$ . Now there exists  $p, q \in \mathbb{Z}$  such that  $u = \sigma^{-p}(\tau(u'))$  and  $v = \sigma^{-q}(\tau(v'))$  but it follows from lemma 5 that  $p = q$  and we can furthermore deduce that these must equal the length of the common postfix of  $\tau(u'[-n, -1])$  and  $\tau(v'[-n, -1])$ . Now repeat this exercise to produce  $u''$  and  $v''$  with  $u''_{[0, \infty[} = v''_{[0, \infty[}$  and  $u''_{[-1]} \neq v''_{[-1]}$  and with  $u' = \sigma^{-r}(\tau(u''))$   $v' = \sigma^{-r}(\tau(v''))$  where  $r$  is the length of the common postfix of  $\tau(u''[-n, -1])$  and  $\tau(v''[-n, -1])$ . Now going from  $(u'', v'')$  to  $(u', v')$  and on to  $(u, v)$  makes the pair of words at index  $[-n, -1]$  change according to the  $rs$  graph and since this is subfixed we have that  $u[-n, -1] = u'[-n, -1]$  and  $v[-n, -1] = v'[-n, -1]$  as desired.  $\square$

*Proof of theorem:* We assume that  $u$  is left special, the right case is, as is often the case, symmetrical. By definition there must exist  $v \in X_\tau$  with  $u_{[0, \infty[} = v_{[0, \infty[}$  and  $u_{[-1]} \neq v_{[-1]}$ . Now let  $n \in \mathbb{N}$  be the least right segregating number, let

$p \in \mathbb{N}_0$  be the length of the common postfix of  $\tau(u[-n, -1])$  and  $\tau(v[-n, -1])$  and let  $r \in \mathbb{N}_0$  be  $|\tau(u[-n, -1])| - p - n$ . Now suppose both  $p$  and  $r$  are nonzero. Then chose

$$g = (u_{[-n-r, -n-1]}, u_{[-n, -1]}, u_{[0, p-1]}).$$

If on the other hand  $r$  is zero and  $p$  nonzero we choose

$$g = (u_{[-n-s-1, -n-2]}, u_{[-n-1, -1]}, u_{[0, p-1]}),$$

where  $s = |\tau(u_{[-n-1]})| - 1$  which is nonzero. If finally  $p$  is zero and  $r$  nonzero we choose

$$g = (u_{[-n-r, -n-1]}, u_{[-n, 0]}, u_{[1, s]}),$$

where  $s = |\tau(u_{[0]})| - 1$  which is nonzero as well. Note that due to primitivity, we cannot have both  $p$  and  $r$  zero. The theorem now follows in each case from iterating lemma 24, making use of the fact that the rl graph is subfixed in the second case and that the ll graph is subfixed in the third case.  $\square$

Let us shortly consider the usefulness of this result: Given a substitution it is often easy to find some special sequences using generators, e.g., any two generators with identical right wings but disagreeing letters in the center complete to left special sequences modulo orbit equivalence. On the other hand, this result tells us that under certain circumstances all special sequences can be obtained in this way. And since the results from the previous section gives us some measure of control over the generators, we are now in a better position to face the special sequences of a substitution. One possible application could be to calculate special sequences and thereby configuration graphs for arbitrary substitutions, but this is already done very well in [1], indeed the present section steals heavily from this source. Instead we shall use our results to produce certain substitutions with desirable properties such as having a particular configuration graph; this is the object of the next section.

## 4 The Zorro Algorithm

### 4.1 Miscellaneous Tools

This subsection contains miscellaneous minor results that are needed in the proof the Zorro Algorithm. While the results are (probably) true, they may appear unmotivated and rather out of context. Do not worry though, all will be clear in due time.

**Lemma 25** *Let  $\tau$  be any primitive substitution. If  $\tau$  has either a left or a right special sequence then it is aperiodic.*

*Proof:* Suppose it has a left special sequence, this provides us with sequences  $x, y \in X_\tau$  with  $x_{[-1]} \neq y_{[-1]}$  and  $x_{[0, \infty[} = y_{[0, \infty[}$ . Assume now that  $\tau$  is periodic, this implies that  $x$  and  $y$  are each periodic, let  $n, m$  be the lengths of their periods. But then both sequences are periodic with periods of length  $nm$  as well which is an obvious contradiction.  $\square$

**Proposition 26** *Let  $\tau$  be any substitution with subfixed ll and rl graphs. We have that*

$$\mathcal{L}_2(\tau) = \{u \in \mathcal{A}_2 \mid \exists a \in \mathcal{A} : u \dashv \tau(a)\} \cup \{u \in \mathcal{A}_2 \mid \exists a \in \mathcal{A} : u \dashv \tau^2(a)\}.$$

*Proof:* Any member of the right hand side is a member of the left hand side by definition. Now let  $u \in \mathcal{L}_2(\tau)$ , by definition we have  $a \in \mathcal{A}$  and  $n \in \mathbb{N}$  with  $u \dashv \tau^n(a)$  and we may chose  $a$  and  $n$  such that  $n$  is minimal. Assume for the sake of contradiction that  $n \geq 3$ . This implies that there can be no letter  $b \dashv \tau^{n-1}(a)$  with  $u \dashv \tau(b)$ , nor any letter  $b \dashv \tau^{n-2}(a)$  with  $u \dashv \tau^2(b)$ . But this again implies that there exist  $v, w \in \mathcal{A}^+$  with  $vw = \tau^{n-2}(a)$  and with  $u = \text{rl}(\tau^2(v))\text{ll}(\tau^2(w))$ . But since the ll and rl graphs are subfixed we have that

$$\text{rl}(\tau^2(v))\text{ll}(\tau^2(w)) = \text{rl}(\tau(v))\text{ll}(\tau(w)),$$

which implies the contradiction  $u \dashv \tau^{n-1}(a)$ .  $\square$

This result can be generalized to word lengths higher than 2. We are, however, more interested in the following corollary:

**Corollary 27** *Let  $\tau$  be any substitution with subfixed ll and rl graphs. Let*

$$W = \{u \in \mathcal{A}_2 \mid \exists a \in \mathcal{A} : u \dashv \tau(a)\}.$$

*We have that*

$$\mathcal{L}_2(\tau) = W \cup \{\text{rl}(\tau(a))\text{ll}(\tau(b)) \mid ab \in W\}.$$

## 4.2 The Theorem and the Algorithm

**Definition 28** *A bipartite graph is said to be undecided if it has the following properties:*

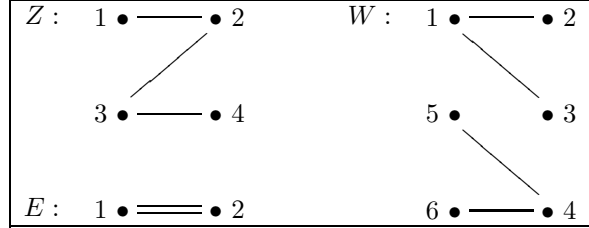
- (i) *There are no lonely vertices, i.e., any vertex has one or more outgoing edges.*
- (ii) *There are no lonely edges, i.e., for any edge there exists another edge with one or both vertices mutual.*
- (iii) *There exists a left vertex with two outgoing edges.*
- (iv) *There exists a right vertex with two outgoing edges.*

We say that a primitive aperiodic substitution *realizes* its configuration graph and in general that a bipartite graph is *realizable* if there exists a primitive, aperiodic substitution realizing it. The following theorem is the conclusion to much of our work:

**Theorem 29** *A bipartite graph is realizable if and only if it is undecided. Indeed, for any bipartite undecided graph the Zorro algorithm described below will compute a primitive, aperiodic substitution realizing it.*

*Proof:* Note initially that by the definitions and results of subsection 1.3 it is immediate that any realizable graph is undecided. To prove the other way round, we shall first state the Zorro algorithm with a few examples and then afterwards consider that it actually produces the desired substitutions.

Consider the following three bipartite graphs:



Now let  $G$  be any bipartite undecided graph. It follows from parts (iii) and (iv) of the definition that  $G$  must contain one or more of the above graphs as a subgraph. The algorithm has three cases corresponding to these three subgraphs, each of these cases proceeds according to the following common recipe but with slightly differing ingredients<sup>1</sup>:

1. The first part simply states an *initial substitution* that realizes the given subgraph. The alphabet has one letter corresponding to each vertex in the subgraph but also contains additional letters that do not correspond to vertices. The following three steps will gradually extend the initial substitution such that the final result realizes  $G$ .
2. Remaining vertices are added now: For each vertex in  $G$  not in the subgraph, we add a new letter to our alphabet. The value of our substitution at these new letters are assigned according to *left* and *right patterns* for left respectively right vertices. To be precise, the value of a new letter corresponding to a left vertex is obtained by postfixing the word produced by the left pattern with the new letter itself, right letters are treated symmetrically.
3. Then the first edges: For each pair of vertices that are presently unconnected but are connected in  $G$  we add the first (possibly only) edge by inserting the two letter word consisting of the two letters corresponding to the left respectively right vertex at the *insertion point* specified as part of the initial substitution.
4. And finally the remaining edges: For any two vertices that are already connected but lack the number of edges present in  $G$ , we add a new letter to our alphabet for each missing edge. The value of the substitution at such a new letter is obtained by taking first the value of the substitution at the letter corresponding to the left vertex minus the rightmost letter, then adding the new letter and finally the value of the substitution at the letter corresponding to the right vertex minus the leftmost letter. All new letters produced in this step are finally added directly as one letter words at the insertion point.

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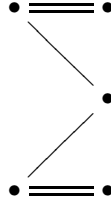
<sup>1</sup>Incidentally, the algorithm is named after the particular shape of the  $Z$  graph, this was the first case solved.

As a start, let us specify the initial substitution with insertion point and left and right patterns in the case of the subgraph  $Z$ , which is the easiest case:

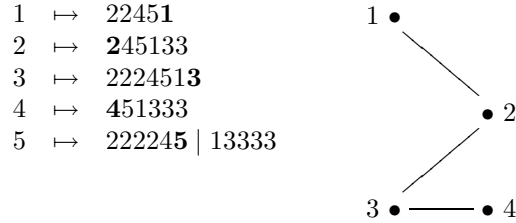
1	$\mapsto$	2245 <b>1</b>	left pattern : $\underbrace{22 \dots 2}_{5,6,7,\dots} 45$
2	$\mapsto$	<b>2</b> 45133	
3	$\mapsto$	222451 <b>3</b>	right pattern : $51 \underbrace{33 \dots 3}_{5,6,7,\dots}$
4	$\mapsto$	<b>4</b> 51333	
5	$\mapsto$	22224 <b>5</b>   13333	

A few words on the notation: The insertion point is specified by a vertical line, in this case in the middle of the value of 5. As a theoretical convenience we have highlighted letters in values letters that are identical to the source letter, this is of no importance when applying the algorithm. The patterns produce words of increasing length, i.e., the first word produced by the left pattern in this case is 2222245, the next 22222245 and so on. Finally note that the letters 1 though 4 corresponds to the vertices of  $Z$  whereas the letter 5 does not correspond to any vertex.

An example is due, indeed we should very much like to realize the following graph:

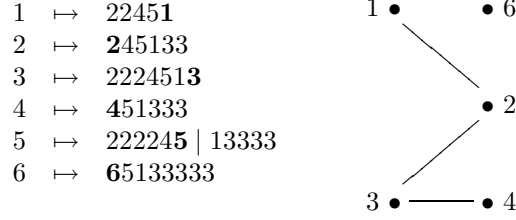


Luckily, it is undecided. Initially we need to identify which of the three graphs that are contained in this graph. As it happens, both the  $Z$  and  $E$  are subgraphs. For didactic reasons we chose to carry on with  $Z$ , but choosing  $E$  would have produced a realizing substitution as well. But then we have an initial substitution and step one of the algorithm is complete and leaves us with the following substitution and its configuration graph:

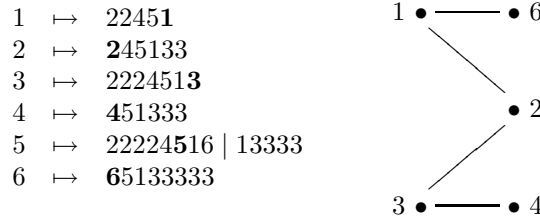


Notice a two things here: The highlighted symbols and the insertion point in the substitution are of course not a part of the substitution but rather theoretically convenient layout, just as the letters labeling the vertices. Also notice that  $Z$  does not occur as subgraph of our graph in an unambiguous way, indeed we could have chosen to let the vertices of  $Z$  coincide with all vertices except the lower right instead. This, like the choice between  $Z$  and  $E$  at step 1, does not matter, all choices will produce realizing, if not necessarily identical, substitutions. As for step two, we need to introduce one more vertex, this is done by adding the letter 6 to our alphabet and assigning it the value 65133333 in accordance

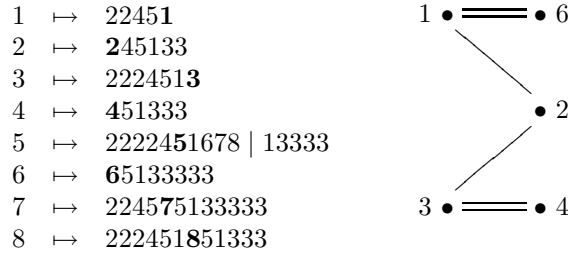
with the right pattern since it is a right vertex. We now have the following substitution and corresponding graph as conclusion to step 2:



Notice about this step that while the substitution above corresponds to the graph in algorithmic terms it does not realize it. This is a slight inconvenience that applies to step two only, essentially it is caused by adding lonely vertices to the original graph and thereby wrecking havoc upon its undecidability. As for step three, we need to add just one edge between the vertices 1 and 6. This is easily done by adding the two letter word 16 at the insertion point:



Finally, we need to add two more edges between already connected vertices: One more between vertices 1 and 6 and the final between the vertices 3 and 4. The first is added by introducing the new letter 7 and assigning it the value 2245 followed by 7 itself followed by 5133333, i.e., the unlikely long value of 224575133333. Similarly the final edge is added by introducing the letter 8 and assigning it the value 222451851333. Both these two new letters are added at the insertion point and the fourth and final step of the algorithm is complete:



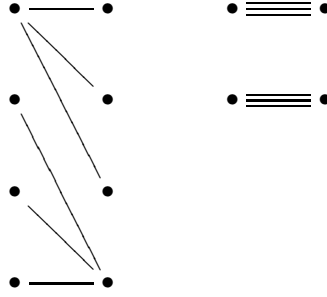
The example concluded, let us now state the initial substitution etc. for the remaining two cases. First the case of the subgraph  $W$ :

1	↦	4237 <b>6</b> 1	
2	↦	<b>2</b> 37651	
3	↦	<b>3</b> 76551	left pattern : 4 $\underbrace{22 \dots 2}_{4,5,6,\dots}$ 37
4	↦	43765551	
5	↦	42237 <b>6</b> 5	right pattern : 76 $\underbrace{55 \dots 5}_{4,5,6,\dots}$ 1
6	↦	422237 <b>6</b>	
7	↦	223 <b>7</b> 47  * 17655	

As hinted by the star next to the insertion point, there is one peculiarity to this case as compared to the two others: All words inserted at the insertion point, whether at step three or four in the algorithm, need to be followed by the letter 7, e.g., if the algorithm tells us to insert the words 53, 8 and 9 at the insertion point, then we need to insert **5378797** and not just their concatenation 5389 as we would in the other two cases. This is caused, in a sense, by the graph  $W$  being disconnected, the symbol 7 works as bridge between the parts. The final case of the subgraph  $E$  completes the definition of the algorithm:

1	$\mapsto$	253425 <b>1</b>	left pattern : 25 <u>11</u> $\cdots$ 1 34
2	$\mapsto$	<b>2</b> 513451	3,4,5,...
3	$\mapsto$	25 <b>3</b> 425 <b>3</b>	right pattern : 34 <u>22</u> $\cdots$ 2 51
4	$\mapsto$	<b>4</b> 513451	3,4,5,...
5	$\mapsto$	2 <b>5</b> 1134   3422 <b>5</b> 1	

As conclusion to our description of the algorithm we provide two more examples, one for each of the graphs  $W$  and  $E$ . We shall not go into the same level of detail as before, but rather just present the desired graphs and then state the results of running the algorithm. The two undecided graphs we would like to realize are:



The first contains the graph  $W$  and running the algorithm gives us the following result:

1	$\mapsto$	42376 <b>1</b>	1 • — • 2
2	$\mapsto$	<b>2</b> 37651	
3	$\mapsto$	<b>3</b> 76551	9 • — • 3
4	$\mapsto$	43765551	
5	$\mapsto$	422376 <b>5</b>	5 • — • 8
6	$\mapsto$	422237 <b>6</b>	
7	$\mapsto$	223 <b>7</b> 4718 <b>7</b> 94 <b>7</b>  * 17655	6 • — • 4
8	$\mapsto$	<b>8</b> 7655551	
9	$\mapsto$	422237 <b>9</b>	

Notice here how the words 18 and 94 are followed by the letter 7 as specified above. The final example gives the following result:



1	$\mapsto$	253425 <b>1</b>	1 • $\equiv$ • 2
2	$\mapsto$	<b>2</b> 513451	
3	$\mapsto$	25 <b>3</b> 425 <b>3</b>	
4	$\mapsto$	<b>4</b> 513 <b>4</b> 51	6 • $\equiv$ • 7
5	$\mapsto$	2 <b>5</b> 113467890   3422 <b>5</b> 1	
6	$\mapsto$	2511134 <b>6</b>	
7	$\mapsto$	<b>7</b> 3422251	
8	$\mapsto$	253425 <b>8</b> 513451	
9	$\mapsto$	2511134 <b>9</b> 3422251	
0	$\mapsto$	2511134 <b>0</b> 3422251	

Having thus stated and exemplified the algorithm it is time to prove that it does indeed produce a primitive, aperiodic substitution realizing a given graph. We shall not go through all painstaking details three times. Instead, we list the properties that need to be verified for all cases and for each of these properties describe the general strategy used to verify it. And we shall, of course, verify a few of these properties in full detail for some of the cases.

The first issue to consider is that of primitivity. This is fundamental to all our workings and luckily it holds easily for all substitutions since they all contain a particular letter with the property that its value contains the entire alphabet and it is itself contained in the value of all letters. This is the letter with the insertion point. Notice that this property also holds after adding additional letters according to step 2 since these are all added at the insertion point in part 3 by part (i) of the definition of an undecided graph. The next basic issue is aperiodicity, but this is easily handled by lemma 25 since left or right special sequences are easily constructed from generators in all the initial substitutions. As an example, the generators (253425, 12, 513451) and (253425, 34, 513451) from case *E* provide us with both left and right special sequences.

Having dealt with the basics, we now check that the produced substitutions are prefix as well as postfix free, this implies that they are segregating with least left and right segregating numbers both 1. With this in mind, we furthermore verify that all the four graphs ll, rl, ls and rs are subfixed. This is where the weird patterns used in step 2 are justified since they oversee that these properties, that hold for the initial substitutions, are maintained through the steps 2, 3 and 4 of the algorithm. Take as an example the case *W*: The initial substitution is easily prefix and postfix free and some checking shows that the four graphs are all subfixed. Now let us add a left vertex as an example of the effects of step 2, we get  $8 \mapsto 42222378$ . On the right hand side the new unique letter 8 protects from trouble. And the left pattern ensures not only that the substitution remains prefix free but also that the ll and in particular the ls graph remain subfixed. Step 3 changes nothing and the letters introduced in step 4 also ends up being compatible with the state of affairs. Taking some time to verify these things also gives some idea of why the produced substitutions tend to be lengthy.

We now have primitive, aperiodic, segregating substitutions with the four graphs ll, rl, ls and rs subfixed. And indeed, we are going strong, these are exactly the prerequisites of theorem 23. The next consideration is to identify the set of basic generators for each substitution and from these verify that the desired graph is actually realized. Let us consider an example to simplify things: Letting

$\tau$  be the initial substitution in the case  $Z$  we easily get by corollary 27 that  $12, 32, 34 \in \mathcal{L}_2(\tau)$  but  $14 \notin \mathcal{L}_2(\tau)$ , which again easily gives us the following basic generators:

$$(22224, 5, 13333), (2245, 12, 45133), (222451, 32, 45133), (222451, 34, 51333).$$

This immediately implies that the orbit classes containing the completions of the three last generators are special and by proposition 16 different. Furthermore, by theorem 23 and corollary 18 these are the only special orbit classes. And by corollary 18 the completions of the second and third are right tail equivalent whereas the completion of the fourth isn't right tail equivalent with any of the others; similarly the completions of the third and fourth are left tail equivalent but the second is excluded. Summing up, we have proved that the initial substitution actually does realize the  $Z$  graph, and in general that, because of our careful preparations above, the configuration graph is easily read off from the set of basic generators.

The general idea is now that any left vertex corresponds to a letter with a value consisting of a unique left part not containing the letter itself followed by the letter. This correspondence is set up in the initial substitution and is maintained through step 2 by the left pattern. The situation is symmetrical for the right vertices. Step 2 does thus not in itself produce any new generators, since the centers of the potential generators are not in the language yet. This setup makes the adding of vertices at step 3 very easy though, just extend the language by adding words at the insertion point, only we have to take some care in the case of case  $W$  not to introduce unwanted generators. Note that by part (ii) of the definition of an undecided graph we are ensured that all edges share a vertex with some other edge, this ensures that the generators we add in this step become special and thus actually figure in the graph. At step 4 we want to add an additional edge between already connected vertices, this is easily done by introducing a new generator with left and right wings corresponding to the vertices but with a new center, and remembering to add it to the language. To satisfactorily verify the algorithm one of course needs to check very carefully that no unwanted two letter words enter the language during the steps 2 through 4, since this would give an undesired edge, we shall refrain from doing this in writing.  $\square$

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